Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021–8928(99)00023-4

J. Appl. Maths Mechs, Vol. 63, No. 2, pp. 161–163, 1999

0021–8928/99/\$—see front matter

SOME PROPERTIES OF A LINEAR-FRACTIONAL TRANSFORMATION†

P. Ya. KOCHINA and N. N. KOCHINA

Moscow

(*Received* 1 *October* 1998)

Seven properties of a linear-fractional analytic function, many of which are also valid in the domain of real variables, are pointed out. In either case, these properties are important for applications to problems of subterranean hydromechanics.‡ © 1999 Elsevier Science Ltd. All rights reserved.

Linear ordinary differential equations of Fuchsian type were already being investigated in the 19th century [1, 2]. An extensive bibliography on these issues may be found in Golubev's book [3], and also in Smirnov's book [4].

For many years, research on this subject has engaged the attention of P. Ya. Kochina (some of her results in this connection were presented in [5, 6]). A. R. Tsitskishvili (see, for example, [7]), E. N. Bereslavskii (see, for example, [8, 9]) and N. N. Kochina (see, for example, [10, 11]).

It should be mentioned that linear-fractional transformations may sometimes be used to transform circular polygons, which play a fundamental role in the theory of analytic equations of Fuchsian type, into simpler domains.

A linear-fractional transformation of the form

$$w = \frac{az+b}{cz+d}, \quad \Delta \equiv ad - bc \neq 0 \tag{1}$$

where, a, b, c and d are complex numbers. Often one uses unimodular transtormations, that is, transformations for which $\Delta = 1$; to obtain such a transformation, one need only divide each of the numbers a, b, c and d by $\pm\sqrt{\Delta}$. Formula (1) shows that

$$dw / dz = \Delta / (cz + d)^2 \tag{2}$$

It is also obvious from (1) that any linear-fractional transformation may be completed to a biunique mapping of the extended plane $\bar{C}$ (i.e. the plane C completed by the point $z = \infty$) onto itself.

We will cite some properties of linear-fractional transformations [1, 3].

1. The linear-fractional transformation (1) maps the extended plane $\bar{C}$ onto itself biuniquely and conformally. Under this mapping, any circle in $\bar{C}$ (i.e. a circle in C or a straight line completed by the point $z = \infty$, as a circle of infinitely large radius) transfers into a circle in $\bar{C}$.

It follows from (2) that a linear-fractional transformation is conformal in the entire z plane except at the point $z = -d/c$, where it has a simple pole. At the point at infinity $z = \infty$, angles are also preserved if (1) is not a linear transformation, that is, if $c \neq 0$ (where by the angle between curves at $z = \infty$ we mean the angle between the curves obtained under the mapping $z = 1/z_1$ at the point $z_1 = 0$). Indeed, substituting $z_1 = 1/z$, one finds that the function $w(z_1)$ is regular at $z_1 = 0$ (provided that $c \neq 0$) and has a non-zero derivative there.

2. A pair of points z, z^* symmetric relative to a circle Q in $\bar{C}$ is mapped onto a pair of points w, w^* symmetric relative to the image of the circle.

3. The cross-ratio (an harmonic ratio) of four points in $\bar{C}$ is invariant under a linear-fractional transformation: if the points z_i are mapped, respectively, onto points w_i (i = 1, 2, 3, 4)

†*Prikl. Mat. Mekh.* Vol. 63, No. 2, pp. 168–171, 1999.

‡These results have already been presented in a brief form, see KOCHINA, P. Ya., BERESLAVSKII, E. N. and KOCHINA, N. N., Analytical theory of linear differential equations of Fuchsian type and some problems of subterranean hydromechanics, Pt 1: Preprint No. 567, Inst. Problem Mekhaniki Ross. Akad. Nauk, Moscow, 1996. Chap. II, Sec. 4 (this section was written by P. Ya. and N. N. Kochina).

$$\frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2} = \frac{w_3 - w_1}{w_3 - w_2} : \frac{w_4 - w_1}{w_4 - w_2} \tag{3}$$

For any given pairwise distinct points z_1, z_2, z_3 and w_1, w_2, w_3 of $\overline{C}$, a linear-fractional transformation exists which maps z_k onto w_k ($k = 1, 2, 3$); this transformation is unique and it may be found from (3) by replacing z throughout by z_4 and w by w_4

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_2)(w_1 - w_3)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_2)(z_1 - z_3)} \tag{4}$$

If one of the points is the point at infinity, the side of formula (4) that contains the point at infinity must be replaced by its limit. If $z_1 = \infty$, or $z_2 = \infty$, or $z_3 = \infty$, one replaces the right-hand side of (4) by

$$-\frac{z_2 - z_3}{z - z_2}, \quad -\frac{z - z_1}{z_1 - z_3}, \quad -\frac{z - z_1}{z - z_2}$$

respectively. Similar changes are necessary on the left-hand side if one of w_1, w_2, w_3 is the point at infinity.

4. The transformation inverse to (1) is

$$z = \frac{-dw + b}{cw - a} \tag{5}$$

that is, it is also a linear-fractional transformation. This transformation, applied after (1), restores any figure to its original position.

The identical transformation

$$w = z \tag{6}$$

is thus the unit transformation.

By repeatedly applying transformation (1) to a unimodular linear-fractional transformation (1), one again obtains a unimodular linear-fractional transformation. Hence it follows that the set of all linear-fractional transformations is a group; the group operation is associative but not necessarily commutative.

Among the subgroups of the group of all linear-fractional transformations, of particular importance, from the viewpoint of the analytical theory of differential equations, are the discrete groups Γ of linear-fractional transformations. Discrete groups Γ of linear-fractional transformations which have an invariant circle γ in $\overline{C}$, common to all transformations of Γ, such that the interior of γ transfers into itself under all transformations of Γ are known as Fuchsian groups.

An example of a Fuchsian group is the modular group, that is, the set of all unimodular linear-fractional transformations (or mappings) (1) in which the coefficients a, b, c and d are real integers. The real axis is invariant under modular linear-fractional transformations [1].

5. If one excludes the identical transformation, it may be deduced that transformation (1) has at most two fixed points ξ_1 and ξ_2 in $\overline{C}$ (such that $w = z = \zeta$). If there are two distinct fixed points $\xi_1 \neq \xi_2$, the family of circles Σ passing through ξ_1 and ξ_2 is mapped onto itself by (1). The family Σ' of all circles orthogonal to the circles of Σ is also mapped onto itself.

If the two fixed points degenerate into one, the family Σ consists of all circles that have a common tangent at ξ_1. Each circle of Σ is mapped onto itself. In this case $a + d = \pm 2$.

6. Function (1) satisfies the following third-order non-linear ordinary differential equation

$$\frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2 = 0 \tag{7}$$

7. Equation (7) is invariant under a linear-fractional transformation

$$z_1 = \frac{\alpha z + \beta}{\gamma z + \delta} \quad (\alpha\delta - \beta\gamma \neq 0)$$

and its general solution contains three independent constants (there are three constants in (1): the quotients of a, b, c and d by any one of them).

Other interesting properties of transformation (1) may be found in [1].

Let us now consider a linear-fractional function in the domain of real variables (x, y)

$$y = f(x) = \frac{ax+b}{cx+d}, \quad \Delta = ad - bc \neq 0 \tag{8}$$

(we now assume that x, y, a, b, c and d are real numbers).

Equation (8) may be written in the form

$$y = y_a - \frac{\Delta}{c^2(x - x_a)}; \quad y_a = \frac{a}{c}, \quad x_a = -\frac{d}{c} \tag{9}$$

Equations (8) and (9) describe an equilateral hyperbola with asymptotes $x = x_a$, $y = y_a$ parallel to the coordinate axes.

We now change from variables (x, y) to variables (ξ, η) according to the following formulae [10, 11]†

$$\xi = \frac{x - x_0}{X - x}, \quad \eta = \frac{y - y_0}{Y - y} \tag{10}$$

where x_0, y_0, X and Y are given constants.

It turns out that, under these transformations, proposed by O. I. Shishorina [12], the linear-fractional function (8) becomes a linear function, namely

$$\eta = B\xi, \quad B = \frac{X - x_a}{x_0 - x_a}$$

Each of the two formulae (10) is also a linear-fractional function—the two functions are known as "simple ratios".

Using transformations (10), in particular, when $x_0 = y_0 = 0$, in the domains $x_0 \leqslant s \leqslant X$, $y_0 \leqslant y \leqslant Y$ it becomes possible to solve many practical problems [12–14]. In particular, certain computed curves obtained in solving the problem of the seepage of water through a rectangular earth-dam [5, 6] may be approximated by empirical formulae—equilateral hyperbolae—using formulae (10) [13].

REFERENCES

1. FORD, L. R., *Automorphic Functions*. McGraw-Hill, New York, 1929.
2. INCE, E. L., *Ordinary Differential Equations*. Longmans, Green and Co., London, 1927.
3. GOLUBEV, V. V., *Lectures on the Analytical Theory of Linear Differential Equations*. Gostekhizdat, Moscow, 1950.
4. SMIRNOV, V. I., *A Course of Higher Mathematics*, Vol. 3, Pt 2. Nauka, Moscow, 1969.
5. POLUBARINOVA-KOCHINA, P. Ya., *Some Problems of the Plane Motion of Groundwater*. Izd. Akad. Nauk SSSR, Moscow, 1942.
6. POLUBARINOVA-KOCHINA, P. Ya., *Theory of the Motion of Groundwater*. Nauka, Moscow, 1952; 2nd edn, 1977.
7. TSITSKISHVILI, A. R., The conformal mapping of a half-plane onto circular polygons. *Dokl. Akad. Nauk SSSR*, 1973, **211**, 2, 300–303.
8. BERESLAVSKII, E. N., The integration in closed form of a certain class of Fuchsian equations and their applications. *Differents. Uravn.*, 1989, **25**, 6, 1048–1050.
9. BERESLAVSKII, E. N. and KOCHINA, P. Ya., Differential equations of Fuchsia class occurring in some problems of fluid mechanics. *Izd. Ross. Akad. Nauk. MZhG*, 1997, 5, 9–17.
10. KOCHINA, P. Ya. and KOCHINA, N. N., *Hydromechanics of Subterranean Water and Problems of Irrigation*. Fizmatlit, Moscow, 1994.
11. KOCHINA, P. Ya. and KOCHINA, N. N., *Problems of Motion with a Free Surface in Subterranean Hydrodynamics*. Redaktsiya Zh. "Uspekhi Fizicheskikh Nauk", Moscow, 1996.
12. SHISHORINA, O. I., The stress concentration around two closely positioned circular apertures. In *Strength Problems in Mechanical Engineering*, No. 9, Izd. Akad. Nauk SSSR, Moscow, 1962, 27–34.
13. KOCHINA, P. Ya. and SHISHORINA, O. I., Linear-fractional transformations and equations of empirical curves. *Prikl. Mat. Mekh.*, 1993, **57**, 1, 215–219.
14. KOCHINA, P. Ya., SAVVINOV, D. D., SHISHORINA, O. I., LARIONOV, V. P., BUNIN, I. Zh. and SAVVINOV, A. S., *Simple Ratios in Nature. Proportionality, Invariance, Similarity*. Nauka, Moscow, 1996.

Translated by D.L.

†See also KOCHINA, P. Ya. and SHISHORINA, O. I., Linear-fractional transformations and their applications. Preprint No. 307. Inst. Problem Mekhaniki, Moscow, 1987.